A anillo conmutativo

a) $u \in \mathcal{U}(A) \Leftrightarrow \exists\, u^{-1} \in A$ t.q $u \cdot u^{-1} = 1$
$u \neq 0$

$\mathcal{U}(\mathbb{Z}_6) = \{ \bar{a} \in \mathbb{Z}_6 \,/\, \boxed{(a,6) = 1} \}$; $\mathbb{Z}_6 = \{ \bar{0}, \underline{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}} \}$

$\bar{0}$ no unidad

$\boxed{\begin{matrix} (1,6) = 1, \\ (5,6) = 1 \end{matrix}}$ $(2,6) = 2$, $(3,6) = 3$, $(4,6) = 2$,

$\mathcal{U}(\mathbb{Z}_6) = \{ \bar{1}, \bar{5} \}$

b) $a\underline{R}b \Leftrightarrow \exists\, u \in \mathcal{U}(A)$ t.q $a = bu$ $[R = \sim]$

R es relación de equivalencia sii es reflexiva, simétrica y transitiva

i) Reflexiva : $\forall a \in A \Rightarrow$ ¿aRa?

$\Downarrow$

¿$\exists u \in \mathcal{U}(A)$ t.q $a = a \cdot u$?

$\boxed{1 \in \mathcal{U}(A)}$ $\boxed{a = a \cdot 1}$

ii) Simétrica : Si $aRb \Rightarrow \boxed{bRa\text{ ?}}$

$\Downarrow$

¿$\exists u \in \mathcal{U}(A)$ t.q $b = au$?

$\exists u_1 \in \mathcal{U}(A)$ t.q $a = bu_1$

$\Downarrow$ $u_1^{-1} \in A$

$au_1^{-1} = (bu_1)u_1^{-1} = b(u_1 u_1^{-1}) = b \cdot 1 = b$

$b = a \cdot u_1^{-1} = au$

Tomo $u = u_1^{-1} \in \mathcal{U}(A)$

$u^{-1} = (u_1^{-1})^{-1} = u_1 \in A$

iii) Transitiva: Si $aRb \wedge bRc \Rightarrow$ $\boxed{¿aRc?}$

$\exists u_1 \in \mathcal{U}(A)$ t.q $a \overset{1)}{=} bu_1$ $\quad$ $\Updownarrow$ $\quad$ ¿$\exists u \in \mathcal{U}(A)$ t.q $a = cu$?

$\exists u_2 \in \mathcal{U}(A)$ t.q $b \overset{2)}{=} cu_2$

$$a \overset{1)}{=} bu_1 = (cu_2)u_1 = c(u_2 u_1) = cu$$

Tomo $u = (u_2 u_1) \in \mathcal{U}(A)$

$$u^{-1} = (u_2 u_1)^{-1} = (u_1^{-1} u_2^{-1}) \in A$$

R es reflexiva, simétrica y transitiva $\Rightarrow$ R es de equivalencia

c) $A/_R = \{\bar a \ / \ a \in A\}$ $\qquad \bar a = \{x \in A \ / \ xRa\}$

$$\mathbb{Z}_6/_R = \{ \underset{\underset{[\bar a]}{\shortparallel}}{\bar{\bar a}} \ / \ \bar a \in \mathbb{Z}_6 \} = \{[\bar 0], [\bar 1], [\bar 2], [\bar 3], [\bar 4], [\bar 5]\}$$

$$[\bar 0] = \{\bar x \in \mathbb{Z}_6 \ / \ \bar x R \bar 0\} = \{\bar x \in \mathbb{Z}_6 \ / \ \bar x = \bar 0 \cdot u, \ u \in \underset{\underset{\{\bar 1, \bar 5\}}{\shortparallel}}{U(\mathbb{Z}_6)}\}$$

$$= \{\bar 0 \cdot u \ / \ u \in \{\bar 1, \bar 5\}\} = \{\underset{\underset{\bar 0}{\shortparallel}}{\bar 0 \cdot \bar 1}, \underset{\underset{\bar 0}{\shortparallel}}{\bar 0 \cdot \bar 5}\} = \{\bar 0\}$$

$$[\bar 1] = \{\bar x \in \mathbb{Z}_6 \ / \ \bar x R \bar 1\} = \{\bar 1 \cdot u \ / \ u \in \{\bar 1, \bar 5\}\} =$$

$$= \{\underset{\underset{\bar 1}{\shortparallel}}{\bar 1 \cdot \bar 1}, \underset{\underset{\bar 5}{\shortparallel}}{\bar 1 \cdot \bar 5}\} = \{\bar 1, \bar 5\}$$

$\boxed{[\bar{2}]} = \{\bar{x} \in \mathbb{Z}_6 \ / \ \bar{x} R \bar{2}\} = \{\bar{2}u \ / \ u \in \{\bar{1}, \bar{5}\}\}$

$= \{\underset{\overset{\shortparallel}{\bar{2}}}{\bar{2} \cdot \bar{1}}, \underset{\overset{\shortparallel}{\overline{10} = \bar{4}}}{\bar{2} \cdot \bar{5}}\} = \{\bar{2}, \bar{4}\}$

$\begin{array}{r|l} 10 & 6 \\ \hline 4 & 1 \end{array}$

$[\bar{3}] = \{\bar{x} \in \mathbb{Z}_6 / \ \bar{x} R \bar{3}\} = \{\bar{3} \cdot u \ / u \in \{\bar{1}, \bar{5}\}\}$

$= \{\underset{\overset{\shortparallel}{\bar{3}}}{\bar{3} \cdot \bar{1}}, \underset{\overset{\shortparallel}{\overline{15} = \bar{3}}}{\bar{3} \cdot \bar{5}}\} = \{\bar{3}\}$

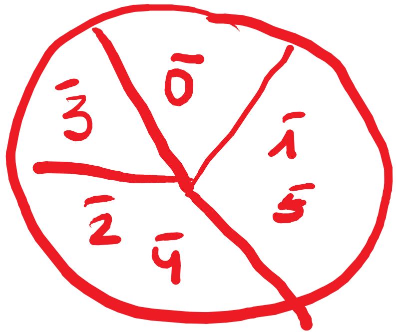
$\begin{array}{r|l} 15 & 6 \\ \hline 3 & 2 \end{array}$

- $[\bar{4}] = [\bar{2}] = \{\bar{2}, \bar{4}\}$ pues $\bar{2} R \bar{4}$
- $[\bar{5}] = [\bar{1}] = \{\bar{1}, \bar{5}\}$ " $\bar{1} R \bar{5}$

$$\mathbb{Z}_6/R = \{[\bar{0}], [\bar{1}], [\bar{2}], [\bar{3}]\}$$

$[\bar{0}] = \{\bar{0}\}$ $\quad$ $[\bar{1}] = \{\bar{1}, \bar{5}\}$, $[\bar{2}] = \{\bar{2}, \bar{4}\}$

$[\bar{3}] = \{\bar{3}\}$



$\mathbb{Z}_6/R$